Open Sets

Characterization of open sets in terms of open spheres.

Defination: Let (X, d) be a metric space. A subset 'G' of 'X' is said to be an open set iff to each $x \in G$, there exists $r > 0$ such that $S(x, r) \subset G$

or.

'G' is an open set if each of its points is the centre of some open sphere contained in 'G'.

Properties

1) On the real line R with the usual metric, an one-pointic set $\{a_n\}$ is not open, for each bounded open interval centered on 'a' contains points not in $\{a\}$.

2) Every open interval $A =]a, b[$ is an open set.

3) The closed interval $B = [a, b]$ is not an open set for any interval containg 'a' or 'b' must contain points outside B.

4) The set $\{\frac{1}{n}, n \in N\}$ is not open since it is not a neighbourhood of any one of its points.

5) The set Q of rational numbers is not open because 'Q' is an enumerable set.

6). A finite non-empty set cannot be open

7) The open rays $]-\infty, a[$ and $]a, \infty[$ are open sets.

Theorem - let (X, d) be a metric space. Prove that ϕ and X are open sets.

Proof: To prove that ϕ is an open set, we have to shat $x \in \phi$ & it must be centre of some open sphere. which is contained in ϕ.
But ϕ is an empty set and has no point in ϕ of which there is not an open sphere.
Thus ϕ vacuowsly satifies the defn of an open set

For each $x \in X$, any open sphere 'x' & radius $\varepsilon > 0$ is contained in X.
Hence 'X' is an open set.

Theorem: Let (X, d) be the discrete metric space. Prove that every subset of X is open.

Proof :- Let G be any subset of X
If $G = \phi$, then G is open.
If $G \neq \phi$, then let $x \in G$ be arbitrary.
Let $r = \frac{1}{2}$ then $S(x, 1/2) = \{x\} \subset G$.

$\therefore$ for any point $x \in G$, we get an open sphere with centre 'x' contained entirely in G. Thus G is open.
Hence every subset of 'X' is open.

Proved

Theorem: Every open sphere in a metric space is an open set.

Proof: Let $S(x, r)$ be an open sphere in the metric space (X, d) with centre $x \in X$ & radius $r > 0$.
Let $y \in S(x, r)$ be any arbitrary.
Let $S(y, \rho)$ be the open sphere with centre 'y' and radius ρ.
If we prove $S(y, \rho) \subseteq S(x, r)$ then we can establish that every open sphere in a metric space is an open set.

We have $d(y,x) < r$

$\Rightarrow r - d(y,x) > 0$

Let $\rho = r - d(y,x) > 0$

Let $z \in S(y,\rho)$ be an arbitrary point.

then $d(y,z) < \rho \Rightarrow d(z,y) < \rho$

Now using M4 we get

$$d(z,x) \le d(z,y) + d(y,x)$$
$$< \rho + (r-\rho) = r$$

$\Rightarrow d(z,x) < \rho$

$\Rightarrow z \in S(x,r)$

$\Rightarrow z \in S(y,\rho) \Rightarrow z \in S(x,r)$

Hence $S(y,\rho) \subseteq S(x,r)$

$\therefore$ For each point of $S(x,r)$ is the centre of some open sphere contained in $S(x,r)$.

$\therefore$ $S(x,r)$ is an open set.

Proved

Theorem : Let (X,d) be a metric space. If $x \in X$ the $N \subseteq X$ is a neighbourhood of 'x' iff there exists an open set containing 'x' and contained in 'N'.

Proof

If Part : Suppose 'N' is a neighbourhood of x. then we have to show that there exists an open set 'G' such that $x \in G \subseteq N$.

Since N is neighbourhood of x, there exists a real no. $r > 0$ such that $x \in S(x, r) \subseteq N$ — (1)

Put $S(x, r) = G$, since an open sphere is also an open set.

Hence 'G' is open & therefore $x \in G \subseteq N$.

<u>Only if Part</u> : let us suppose there exists an open set G such that

$x \in G \subseteq N$. — (2)

Now, we have to prove that 'N' is a neighbourhood of 'x'.

Since 'G' is an open set & $x \in G$, then by defn of open set, there exists a real no. $r > 0$ such that $S(x, r) \subseteq G$.

Hence (2) $\Rightarrow$ $S(x, r) \subseteq N$

$\therefore$ By definition 'N' is a neighbourhood of 'x'.

<u>Proved</u>